AN ASYMPTOTIC METHOD FOR STUDYING NONEQUILIBRIUM SYSTEMS

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 50–56, 1966

The following is a well-known problem of statistical physics: can a dynamic system of oscillators with nonlinear coupling be described approximately by statistical laws? This problem was studied for the first time by Fermi, Ulam, and Pasta [1] for the following system of equations describing coupled oscillators:

$$x_i^{\cdot\cdot} = x_{i+1} - 2x_i + x_{i-1} + \alpha\,[(x_{i+1} - x_i)^2 - (x_i - x_{i-1})^2], \quad (i = 1, \ldots, N;\ \alpha < 1)\,. \tag{0.1}$$

The numerical solution of system (0.1) did not lead, as expected, to the appearance of any statistical properties. The negative result led to further investigations [2, 3]. System (0.1) was solved either numerically or with the aid of perturbation theory, but stochasticity was not observed. The behavior of a nonlinear oscillator under the action of an external force depending in a given manner on time has also been studied [4, 5]. (See also B. V. Chirikov, Dissertation, Novosibirsk, 1959.) A criterion which would indicate approximate stochasticity of the system was found.

An asymptotic method is proposed below for investigating a system of coupled oscillators which makes it possible, with certain definite restrictions, to decide whether statistical methods can be applied to a system and what the criterion of this possibility should be.

In the first section a method is constructed which permits obtaining asymptotic solutions for a system of linearly coupled oscillators, all of whose parameters depend slowly on time. In the second section, this method is applied to a system of two oscillators with nonlinear coupling.

§1. Description of method. A system of linearly coupled oscillators is described by the equations

$$x_i^{\cdot\cdot} + \omega_i^2(t)\,x_i = \sum_{k \neq i} \alpha_{ik}(t)\,x_k \qquad (i = 1, \ldots, N) \tag{1.1}$$

where the frequencies ω_i and the coupling parameters α_{ik} vary slowly with time (this may be due, for example, to external fields). Without any loss of generality, we shall consider, for the sake of simplicity, that the characteristic times T of variation of the quantities α_{ik}, ω_k are the same. We write the slowness condition in the form

$$\frac{1}{T} \ll \frac{d}{dt}(\ln x_i) \qquad (i = 1, 2, \ldots, N). \tag{1.2}$$

We shall also consider the natural frequencies ω_i and the coupling frequencies α_{ik} as analytic functions of t for all real t and in a sufficiently large region of imaginary t (the last condition will be made specific). When ω_i and α_{ik} are not time-dependent, the Lagrangian of the system (1.1), which is a quadratic form in x_k, can be reduced to diagonal form. In this case, the solution of system (1.1) is represented in the form of N normal oscillations which are not coupled with each other and have normal frequencies. In the case under consideration, this transformation can not be performed by a single method over the entire real axis t as the transformation matrix becomes singular at points on the plane of the complex variable t where the characteristic numbers of system (1.1) coincide; this means coincidence of the squares of the frequencies of Ω_k^2-normal oscillations and can lead to redistribution of energy among the normal modes. We shall call the special points where Ω_k^2 coincide for different k resonance points. We shall seek solutions of system (1.1) in the form of asymptotic series [6]:

$$y_i = \Pi_i(t)\exp\{i\int^t \Omega_i(\tau)\,d\tau\}(1 + \ldots)\,,$$
$$y_i^* = \Pi_i(t)\exp\{-i\int^t \Omega_i(\tau)\,d\tau\}(1 + \ldots)\,,$$
$$\left(\Pi_i(t) = \frac{1}{\sqrt{\Omega_i(t)}}\right). \tag{1.3}$$

Here $\Omega_i(t)$ are the roots of the characteristic equation which are determined just as in the case of constant ω_i, α_{ik}.

The series (1.3) are expanded in terms of a small parameter following from (1.2) (refer, for example, to reference [7]); however, it will be sufficient to limit ourselves to just the principal terms of the asymptotic expansions written out in (1.3). When $t < t_-$, let the solution be given in the form

$$Y_- = \sum_{i=1}^{N} (A_i y_i + A_i^* y_i^*) \tag{1.4}$$

where A_i are arbitrary coefficients; we are required to find the solutions when $t > t_+$:

$$Y_+ = \sum_{i=1}^{N} (B_i y_i + A_i^* y_i^*)\,. \tag{1.5}$$

The answer will not be trivial if there is a singular point t_0 (more precisely, Re t_0) in the interval (t_-, t_+) on the real axis. As is well known, this is connected with the presence of Stokes lines of the asymptotic solutions (1.3). The coefficients of the solutions A_i, A_i^* change abruptly on transition from one Stokes line to another.

The relation between the coefficients A_i, A_i^* and B_i, B_i^* for a system of two oscillators was obtained in reference [6]. In this case, we have two pairs of singular points (O_1, O_2) (Fig. 1) in which $\Omega_1 = \pm\Omega_2$, respectively. Due to the realness of the coefficients of (1.1), we have $t_{O_1} = t_{O_1}^*$, $t_{O_1'} = t_{O_2'}^*$. The dashed lines are those on which the expressions $\Omega_1 \pm \Omega_2$ are purely imaginary; however, $(\Omega_1 \pm \Omega_2)^2$ have simple zeros at the corresponding points. We shall represent the result of reference [6] obtained by a method analogous to Zwaan's method in the form

$$B = M_1A, \quad B = M_2A, \quad B = MA; \tag{1.6}$$

$$A \equiv \begin{pmatrix} A_1 \\ A_2 \\ A_2^* \\ A_1^* \end{pmatrix}, \quad B \equiv \begin{pmatrix} B_1 \\ B_2 \\ B_2^* \\ B_1^* \end{pmatrix},$$

$$M_1 \equiv i\begin{pmatrix} \cos\varphi_1 & -\sin\varphi_1 & 0 & 0 \\ \sin\varphi_1 & \cos\varphi_1 & 0 & 0 \\ 0 & 0 & \cos\varphi_1 & \sin\varphi_1 \\ 0 & 0 & -\sin\varphi_1 & \cos\varphi_1 \end{pmatrix}, \tag{1.7}$$

$$M_2 \equiv \begin{pmatrix} \cos\varphi_2 & 0 & \sin\varphi_2 & 0 \\ 0 & \cos\varphi_2 & 0 & -\sin\varphi_2 \\ -\sin\varphi_2 & 0 & \cos\varphi_2 & 0 \\ 0 & \sin\varphi_2 & 0 & \cos\varphi_2 \end{pmatrix}, \qquad M = M_1M_2^+;$$

$$\sin^2\varphi_1 \equiv \exp\left\{\frac{i}{2}\oint_{L_1}(\Omega_1 - \Omega_2)\,d\tau\right\},$$

$$\sin^2\varphi_2 \equiv \exp\left\{\frac{i}{2}\oint_{L_1}(\Omega_1 + \Omega_2)\,d\tau\right\}. \tag{1.8}$$

The contours L_1, L_2 are shown in Fig. 1. The integrals in (1.8) are purely imaginary and their magnitude is determined by the distance of points O_1, $O_1^!$ from the real axis. The matrices M_1, M_2, M determine the transformation of the coefficients A_i, A_i^* when passing through resonances of the type O_1, $O_1^!$ and $O_1 + O_1^!$, respectively. The following is a valuable property of all matrices of (1.7):

$$\begin{aligned}(AA^+) &= |A_1|^2 + |A_2|^2 = (BB^+) = \mathrm{inv} \equiv I, \\ A^+ &= (A_1^*, A_2^*, A_2, A_1), \qquad B^+ = (B_1^*, B_2^*, B_2B_1).\end{aligned} \tag{1.9}$$

The invariant (1.9) has the following interesting physical significance. We introduce for consideration the amplitudes of the oscillations A_i° which, according to (1.3) are connected with A_i in the following manner:

$$A_i = A_i^\circ\sqrt{\Omega_i}. \tag{1.10}$$

Then

$$\begin{aligned} I = |A_1^\circ|^2\Omega_1 + |A_2^\circ|^2\Omega_2 = E_1/\Omega_1 + E_2/\Omega_2, \\ E_i = \Omega_i^2|A_i^\circ|^2, \end{aligned} \tag{1.11}$$

where E_i is the energy of the i-th mode. Thus, when passing through resonance points, a magnitude is maintained which is equal to the sum of the formal expressions for the adiabatic invariants of each of the oscillators separately, the latter changing abruptly upon transition. Formulas (1.6) and (1.7) define the law connecting the adiabatic invariants of each degree of freedom in a collision, and (1.11) determine the integral of motion of the entire system. In the case of system (1.1) of N oscillators, the points of coincidence of an arbitrary number of characteristic roots will be the singular points. We note that coincidence is not necessarily to be understood in the literal sense —simple proximity of singular points is sufficient.

We shall begin by considering paired resonances (that is, points where some pairs of normal frequencies coincide). In accordance with (1.7) and (1.8), we have

$$B = \Big(\prod_{i,k} M_{ik}\Big)A \tag{1.12}$$

where A and B are column vectors having the respective components $(A_1, \ldots, A_N, A_N^*, \ldots, A_1^*)$, $(B_1, \ldots, B_N, B_N^*, \ldots, B_1^*)$, and the matrix M_{ik} has all elements $m_{\alpha\beta}$ equal to zero with the exception of the following:

$$m_{\alpha\alpha} = 1,$$

$$m_{ik} = -m_{ki} = -\sin\varphi_{ik} = m_{N+k,N+i} = -m_{N+i,N+k},$$

$$m_{ii} = m_{kk} = \cos\varphi_{ik} = m_{N+i,N+i} = m_{N+k,N+k},$$

$$\sin^2\varphi_{ik} = \exp\Big(\frac{i}{2}\oint_{L_{ik}}(\Omega_i - \Omega_k)\,d\tau\Big). \tag{1.13}$$

The contours L_{ik} are analogous to the contour L_i of Fig. 1. The product in (1.12) is taken in the order of sequence of resonance points with motion along the real t axis from $t < t_-$ to $t > t_+$. If at this time, we encounter points at which $\Omega_i + \Omega_k = 0$, then they are considered analogously with replacement of the elements of (1.13) with corresponding ones from (1.7) for the matrix M_2. Taking the unitary nature of M_{ik} into consideration, we have the invariant

$$I = (BB^+) = (AA^+) = \sum_{i=1}^N |B_i|^2 = \sum_{i=1}^N \frac{E_i}{\Omega_i}. \tag{1.14}$$

Now, we shall go on to consider points where three normal frequencies coincide $\Omega_i = \Omega_k = \Omega_l$ ("triple resonance"). Here we also have the case in which the points of paired resonances $\Omega_i = \Omega_k$ and $\Omega_k = \Omega_l$ are close to each other.

We shall show that this case can be reduced to paired resonance. We write $y_{i,kl}$ in the form

$$\frac{y_i}{\Pi_i(t)} \equiv a\exp\left\{\frac{i}{2}\int^t(\Omega_i + \Omega_k)\,d\tau\right\}\exp\left\{\frac{i}{2}\int^t(\Omega_i - \Omega_k)\,d\tau\right\} + (1-a)\exp\left\{\frac{i}{2}\int^t(\Omega_k + \Omega_i)\,d\tau\right\}\exp\left\{\frac{i}{2}\int^t(\Omega_k - \Omega_i)\,d\tau\right\},$$

$$\frac{y_k}{\Pi_k(t)} \equiv (1-b)\exp\left\{\frac{i}{2}\int^t(\Omega_k + \Omega_i)\,d\tau\right\}\times \exp\left\{\frac{i}{2}\int^t(\Omega_k - \Omega_i)\,d\tau\right\} + b\exp\left\{\frac{i}{2}\int^t(\Omega_k + \Omega_i)\,d\tau\right\}\exp\left\{\frac{i}{2}\int^t(\Omega_k - \Omega_i)\,d\tau\right\} - 9, \tag{1.15}$$

$$\frac{y_l}{\Pi_l(t)} \equiv c\exp\left\{\frac{i}{2}\int^t(\Omega_l + \Omega_i)\,d\tau\right\}\exp\left\{\frac{i}{2}\int^t(\Omega_l - \Omega_i)\,d\tau\right\} + (1-c)\exp\left\{\frac{i}{2}\int^t(\Omega_l + \Omega_k)\,d\tau\right\}\exp\left\{\frac{i}{2}\int^t(\Omega_l - \Omega_k)\,d\tau\right\},$$

where a, b, c are as yet arbitrary numbers. In a like manner, we also write $y_{i,kl}^*$. For the sake of definiteness, we consider that $\Omega_i > \Omega_k > \Omega_l$. The values of the quantities Ω_i, Ω_k, and Ω_l coincide only at the point

(O_1, O_2 in Fig. 2) and not in a finite interval, thus the lines on which $(\Omega_i - \Omega_k)$, $(\Omega_i - \Omega_l)$, $(\Omega_k - \Omega_l)$ are purely imaginary do not coincide and diverge at a sufficiently great distance from O_1, O_2 (Fig. 2). When going around points O_1, O_2 in the complex t plane, this makes it possible to make use of the very same rules for changes in the coefficients for the corresponding exponents as in the case of paired resonance. It now remains for us to determine the arbitrary numbers a, b, c. Since the true solution of system (1.1) is an analytic function in the region under consideration, it is necessary to require that the asymptotic solution at point K (Fig. 2) coincide with that obtained as a result of going around an arbitrary contour L with return to point K. This immediately leads to the following constraints on a, b, c:

$$\frac{a}{b} = \frac{1-a}{c} = \frac{1-b}{1-c}, \qquad a = b = c .$$

From this

$$a = b = c = {}^1/_2 .$$

Cases in which an arbitrary number of normal frequencies intersect at a singular point also reduce to paired resonances in an analogous manner.

The theory considered here makes it possible to take account of internal resonances in a system of coupled oscillators quite simply by making use of a linear method. Passage through resonances can be interpreted as collisions of oscillators. Collisions are accompanied, on the one hand, by redistribution of energy and, on the other, by phase changes in the oscillators. All these changes can be found directly from matrices for transforming A on passage through a resonance. This makes it possible, in particular, to determine whether the law of phase changes in oscillators is near-random. The latter depends on the parameters of the problem and the nature of the distribution of singular points in the complex t plane, that is, on the form of functions $\omega_i(t)$ and $\alpha_{ik}(t)$.

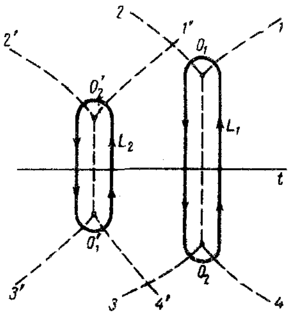

Fig. 1

§2. **The statistical properties of a system of two oscillators with nonlinear coupling.** In this section we shall show how the method developed here can be used to answer the question—under what conditions do properties close to statistical appear in a dynamic system of coupled oscillators? We shall consider a sufficiently simple case of two oscillators with the Lagrange function

$$L = {}^1/_2\,(\dot{x}^2 + \dot{y}^2) - {}^1/_2\omega_1^2 x^2 - {}^1/_2\omega_2^2 y^2 + {}^1/_2\alpha\, x^2 y^2 \quad (2.1)$$

where ω_1, ω_2, α are constants which are not time-dependent. The equations of motion are

$$\ddot{x} + (\omega_1^2 - \alpha y^2)x = 0, \quad \ddot{y} + (\omega_2^2 - \alpha x^2)y = 0. \quad (2.2)$$

We shall impose the following system of inequalities on the system parameters:

$$\omega_1^2 \gg \alpha A_2^{\circ 2} \gg \omega_2^2 \gg \alpha A_1^{\circ 2}. \qquad (2.3)$$

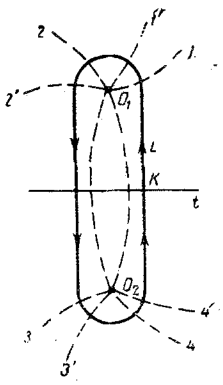

Fig. 2

Here A_1°, A_2° are the respective amplitudes of the x and y oscillations. The first and last inequalities in (2.3) denote the weakness of the nonlinear coupling. As a linear approximation to (2.2), we take

$$\begin{gathered} \ddot{x}_0 + \omega_1^2 x_0 = 0, \quad \ddot{y}_0 + \omega_2^2 y = 0, \\ x_0 = A_1 e^{i\omega_1 t} + A_1^* e^{-i\omega_1 t}, \quad y_0 = A_2^\circ e^{i\omega_2 t} + A_2^{\circ *} e^{-i\omega_2 t}. \end{gathered} \quad (2.4)$$

We shall have the following approximation:

$$\ddot{x} + (\omega_1^2 - \alpha y_0^2)x = 0, \quad \ddot{y} + (\omega_2^2 - \alpha x_0^2)y = 0. \quad (2.5)$$

It can be seen from the equations for y in (2.5) that y can be represented in the form of a sum of rapidly and slowly varying components; however, according to (2.3), the rapidly oscillating component y* is small compared with the slowly varying component Y

$$y^* \sim \frac{\alpha\,|A_1^\circ|^2}{\omega_1^2}\,Y.$$

We write the equation for Y in the usual manner

$$\ddot{Y} + (\omega_2^2 - {}^1/_2\alpha\,|A_1^\circ|^2)Y \approx 0 . \qquad (2.6)$$

We employ the notation

$$\Omega_1^2(t) = \omega_1^2 - \alpha y_0^2(t), \quad \Omega_2^2 = \omega_2^2 - {}^1/_2\alpha\,|A_1^\circ|^2. \quad (2.7)$$

The quantities Ω_1, Ω_2 are the frequencies of normal oscillations for x from (2.5) and for Y. It is not difficult to see that $\Omega_1(t)$ satisfies the conditions of slowness of variation. Thus, the principal terms of the asymptotic solutions of system (2.2) are of the form

$$x = A_1^\circ \exp\left\{i\int^t \Omega_1(\tau)\,d\tau\right\} + A_1^{\circ *}\exp\left\{-i\int^t \Omega_1(\tau)\,d\tau\right\},$$
$$y \approx A_2^\circ \exp\left\{i\int^t \Omega_2\,d\tau\right\} + A_2^{\circ *}\exp\left\{-i\int^t \Omega_2\,d\tau\right\}. \tag{2.8}$$

Here we note a fact which is of importance in utilizing expressions (2.3). The frequencies Ω_1, Ω_2 depend on A_2°, A_1°, respectively. Thus, after passing through resonance, the normal frequencies also vary abruptly. In other words, the closest resonance point is determined for given A_1°, A_2°; after passing through a singular point, the frequencies are corrected and the next resonance is found through them, and so on (this operation is shown conventionally in Fig. 3). This is a consequence of the nonlinearity of the initial equations. Let

$$A_1^\circ = \frac{A_1}{\sqrt{\Omega_1}} = |A_1^\circ|\,e^{i\psi_1}, \qquad A_2^\circ = \frac{A_2}{\sqrt{\Omega_2}} = |A_2^\circ|\,e^{i\psi_2} \tag{2.9}$$

where ψ_1, ψ_2 are the phases of the normal modes immediately before resonance (at point P in Fig. 3). After "collision" at point O_1, we have, according to (1.6)

$$|B_1|^2 = |A_1|^2\{\cos^2\psi_1\cos^2(\varphi_1-\varphi_2) + \sin^2\psi_1\cos^2(\varphi_1+\varphi_2)\} + |A_2|^2\{\cos^2\psi_2(\sin^2(\varphi_1-\varphi_2) + \sin^2\psi_2\sin^2(\varphi_1+\varphi_2)\} - |A_1||A_2|\{\cos\psi_1\cos\psi_2\sin 2(\varphi_1-\varphi_2) - \sin\psi_1\sin\psi_2\sin 2(\varphi_1+\varphi_2)\},$$

$$|B_2|^2 = |A_1|^2\{\cos^2\psi_1\sin^2(\varphi_1-\varphi_2) + \sin^2\psi_1\sin^2(\varphi_1+\varphi_2)\} + |A_2|^2\{\cos^2\psi_2\cos^2(\varphi_1-\varphi_2) + \sin^2\psi_2\cos^2(\varphi_1+\varphi_2)\} + |A_1||A_2|\{\cos\psi_1\cos\psi_2\sin 2(\varphi_1-\varphi_2) - \sin\psi_1\sin\psi_2\sin 2(\varphi_1+\varphi_2)\},$$

$$\operatorname{ctg}\psi_1^\circ = \frac{|A_1|\cos\psi_1\cos(\varphi_1-\varphi_2) - |A_2|\cos\psi_2\sin(\varphi_1-\varphi_2)}{|A_1|\sin\psi_1\cos(\varphi_1+\varphi_2) - |A_2|\sin\psi_2\sin(\varphi_1+\varphi_2)},$$
$$\operatorname{ctg}\psi_2^\circ = \frac{|A_1|\cos\psi_1\sin(\varphi_1-\varphi_2) + |A_2|\cos\psi_2\cos(\varphi_1-\varphi_2)}{|A_1|\sin\psi_1\sin(\varphi_1+\varphi_2) + |A_2|\sin\psi_2\cos(\varphi_1+\varphi_2)}.$$

Here ψ_1°, ψ_2° are determined by the relationships $|B_1| = B_1 e^{i\psi_1^\circ}$, $|B_2| = B_2 e^{i\psi_2^\circ}$.

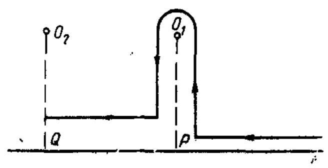

Fig. 3

Formulas (2.10) make it possible to determine the position of the following resonance O_2 (Fig. 3) and to find the phases of the normal modes ψ_1', ψ_2' at point Q before resonance O_2. We are to compare $\psi_{1,2}$ and $\psi_{1,2}'$ which are connected by the definite functional relationship

$$\psi_1' \equiv 2\pi\{\Psi_1[\psi_1,\psi_2]\}, \qquad \psi_2' \equiv 2\pi\{\Psi_2[\psi_1,\psi_2]\}, \tag{2.11}$$

where $\{\Psi\}$ denotes the fractional part of Ψ. The quantity $\Psi[\psi_1, \psi_2]$ takes account of the phase jump (2.10) due to passing through the resonance O_1 and a phase advance of the type

$$\int_P^Q \Omega\,d\tau$$

between two collisions. Let us consider the correlation* of phases $\psi_{1,2}'$

$$R = \left(\int_0^{2\pi} d\psi_1\int_0^{2\pi} d\psi_2(\psi_1'-\pi)(\psi_2'-\pi)\right)\left(\int_0^{2\pi}(\psi-\pi)^2\,d\psi\right)^{-1}. \tag{2.12}$$

A one-dimensional integral of the type (2.12) was computed in reference [5]. Let

$$\Psi_1[\psi_1,\psi_2] = k_{11}\psi_1 + k_{12}\psi_2,$$
$$\Psi_2[\psi_1,\psi_2] = k_{21}\psi_1 + k_{22}\psi_2. \tag{2.13}$$

Then, it is not difficult to show that

$$R \sim \frac{1}{\Delta}, \qquad \Delta = k_{11}k_{22} - k_{21}k_{12} \qquad \text{for } \Delta \gg 1. \tag{2.14}$$

In a more general case

$$R \sim \frac{1}{\langle D\rangle}, \qquad \langle D\rangle = \left\langle\frac{\partial(\Psi_1,\Psi_2)}{\partial(\psi_1,\psi_2)}\right\rangle \gg 1. \tag{2.15}$$

Here the angular brackets denote the average value of the Jacobian D. Thus, if $\langle D\rangle \gg 1$, then the phase correlation can be considered equal to zero with accuracy to $\langle D\rangle^{-1}$, and the phases themselves as random.** From (2.7), the condition of coincidence of frequencies yields $\cos\varphi_{1,2} \sim 1$, $\sin\varphi_{1,2} \sim 1$.

The distance on the real t axis between resonances is about $1/\omega_2$. Omitting the calculations for the example under consideration, and making use of formulas (2.7), (2.10), and (2.15), we obtain the randomicity criterion for oscillator phases

$$\langle D\rangle \sim \sin\varphi\,\frac{\alpha|A_1^\circ|^2}{\omega_2^2}\,\frac{|A_2^\circ|^2\alpha}{\omega_1^2}\,\frac{\omega_1}{\omega_2} \gg 1,$$
$$\varphi = \max(\varphi_1, \varphi_2). \tag{2.16}$$

When criterion (2.16) is satisfied, the oscillator phases are "distributed" as a result of collisions, and system (2.1) can be approximately described with the aid of statistical laws.

*In writing R, ergodicity, which apparently exists in the example under consideration, is assumed.

** B. V. Chirikov directed the author's attention to criterion (2.15).

We shall now discuss the statistical properties of the system under consideration. The latter is described by the Lagrangian (2.1) from which the rapidly oscillating component y* is eliminated. The "contracted" system has two degrees of freedom—x and Y. During collisions, the amplitude of Y and, consequently, the amplitude of y* vary abruptly. Thus, the energy of system (X, Y) will not be an integral of motion. According to (2.12), the integral of motion will be the action

$$I = I_1 + I_2 = \frac{E_x}{\Omega_1} + \frac{E_y}{\Omega_2}. \qquad (2.17)$$

Here E_x, E_y are the respective energies of x and Y oscillations. We can write the kinetic equation for the process describing the relaxation of the system in the usual manner. We shall discuss only the state to which the system relaxes and show that the state in which

$$\langle I_1 \rangle = \langle I_2 \rangle = {}^1/_2 I \qquad (2.18)$$

is the equilibrium state.

Here the angular brackets $\langle \ldots \rangle$ denote averaging over all possible states of the subsystem. Let us consider the state in which

$$|A_1|^2 = |A_2|^2 = |A|^2. \qquad (2.19)$$

According to (1.6), (1.7), after a collision we have

$$|B_1|^2 = |A|^2 - (A_1 A_2^* + A_1^* A_2) \sin\varphi \cos\varphi,$$
$$|B_2|^2 = |A|^2 + (A_1 A_2^* + A_1^* A_2) \sin\varphi \cos\varphi. \qquad (2.20)$$

After averaging (2.20) over all states and taking account of the randomicity of the phases of amplitudes A_1, A_2, we obtain $\langle |B_1|^2 \rangle = \langle |B_2|^2 \rangle$, which immediately leads to (2.18). Thus, we arrive at the equidistribution with respect to actions of each degree of freedom in the equilibrium state.

In conclusion, the author thanks R. Z. Sagdeev and B. V. Chirikov for valuable discussion and criticism.

REFERENCES

1. E. Fermi, J. Pasta, and S. Ulam, "Studies of nonlinear problems," Los Alamos Scientific Report, LA-1940, 1955.

2. J. Ford, "Equipartition of energy for nonlinear systems," J. Math. Phys., vol. 2, p. 387, 1961.

3. T. Ford and J. Waters, "Computer studies of energy sharing and ergodicity for nonlinear oscillator systems," J. Math. Phys., vol. 4, p. 1293, 1963.

4. E. A. Jackson, "Nonlinear coupled oscillators," I, II. Journ. Math. Phys., vol. 4, p. 551, 686, 1963.

5. B. V. Chirikov, Atomnaya energiya, no. 6, 630, 1959.

6. G. M. Zaslavskii and B. V. Chirikov, "On the mechanism of the Fermi acceleration in the one-dimensional case," DAN SSSR, vol. 159, no. 2, 1964.

7. G. M. Zaslavskii and S. S. Moiseev, "Coupled oscillators in the adiabatic approximation," DAN SSSR, vol. 161, no. 2, 318, 1965.

8. G. M. Zaslavskii, Lectures on the Application of the VKB Method in Physics [in Russian], Izd. Novosibirsk, un-ta, 1965.

9. A. Zwaan, Intensitäten im Ca-Funkenspectrum, Utrecht., 1929.

12 November 1965 Novosibirsk